Riccati Equation: $y' = a_2(x)y^2 + a_1(x)y + a_0(x)$

a_2, a_1 and a_0 are functions depending on x.

__Theorem__: If $f(x)$ satisfies a Riccati equation, then the general solution of the Riccati equation is $\underline{\underline{y}} = f(x) + \frac{1}{\underline{\underline{z}}}$, where z satisfies a linear first order equation. (ODE)

Examples: 1 Consider $y' = -y^2 + \frac{2}{x^2}$

a- Verify that $f(x) = -\frac{1}{x}$ satisfies the given equation

b- Solve the given equation.

sln: a- $f' = -f^2 + \frac{2}{x^2}$ $\qquad f = -\frac{1}{x}$

$$f' = \frac{1}{x^2} \checkmark \qquad -f^2 + \frac{2}{x^2} = -\left(-\frac{1}{x}\right)^2 + \frac{2}{x^2} = -\frac{1}{x^2} + \frac{2}{x^2} = \frac{1}{x^2} \checkmark$$

f satisfies the eqn.

b- $y = f(x) + \frac{1}{z} = -\frac{1}{x} + \frac{1}{z}$ put this in the eqn.

$$y' = -y^2 + \frac{2}{x^2} \Rightarrow$$

$$\frac{1}{x^2} - \frac{z'}{z^2} = -\left(-\frac{1}{x} + \frac{1}{z}\right)^2 + \frac{2}{x^2}$$

$$\frac{1}{x^2} - \frac{z'}{z^2} = -\left(\frac{1}{x^2} + \frac{1}{z^2} - \frac{2}{xz}\right) + \frac{2}{x^2}$$

$$\cancel{\frac{1}{x^2}} - \frac{z'}{z^2} = \cancel{-\frac{1}{x^2}} - \frac{1}{z^2} + \frac{2}{xz} + \cancel{\frac{2}{x^2}}$$

$$\frac{-z'}{z^2} = -\frac{1}{z^2} + \frac{2}{xz} \qquad \text{Multiply by } -z^2$$

$z' + \frac{2}{x}z = 1 \rightarrow$ linear ODE

$z^2 \quad - \quad z^2 \quad xz$

$$z' = 1 - \frac{2}{x} z \implies z' + \frac{2}{x} z = 1 \rightarrow \text{linear ODE in } z.$$

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2 \text{ is an I.F.}$$

Multiply the last eqn by x^2

$$x^2 z' + 2xz = x^2 \implies (x^2 z)' = x^2 \implies x^2 z = \int x^2 dx$$

$$x^2 z = \frac{x^3}{3} + c \implies z = \frac{\frac{x^3}{3} + c}{x^2}$$

$$y = -\frac{1}{x} + \frac{1}{z} = -\frac{1}{x} + \frac{x^2}{\frac{x^3}{3} + c} = -\frac{1}{x} + \frac{3x^2}{x^3 + 3c}$$

② Consider $y' = \frac{1}{x^3} y^2 + \frac{1}{x} y - \frac{1}{x}$

a) Verify that $f(x) = x$ satisfies the given eqn.

b) Solve the given eqn.

c) the IVP $y' = \frac{1}{x^3} y^2 + \frac{1}{x} y - \frac{1}{x}$; $y(1) = -1$.

a) $f' = \frac{1}{x^3} f^2 + \frac{1}{x} f - \frac{1}{x}$

$f(x) = x \qquad f' = 1$ $=$

$$\frac{1}{x^3} f^2 + \frac{1}{x} f - \frac{1}{x} = \frac{1}{x^3} x^2 + \frac{1}{x} \cdot x - \frac{1}{x} = \frac{1}{x} + 1 - \frac{1}{x} = 1$$

f satisfies the eqn.

b) $y = x + \frac{1}{z} \qquad y' = \frac{1}{x^3} y^2 + \frac{1}{x} y - \frac{1}{x}$

$$1 - \frac{z'}{z^2} = \frac{1}{x^3}\left(x + \frac{1}{z}\right)^2 + \frac{1}{x}\left(x + \frac{1}{z}\right) - \frac{1}{x}$$

$$1 - \frac{z'}{z^2} = \frac{1}{x^3}\left(x^2 + \frac{1}{z^2} + \frac{2x}{z}\right) + 1 + \frac{1}{xz} - \frac{1}{x}$$

$$\cancel{1} - \frac{z'}{z^2} = \cancel{\frac{1}{x}} + \frac{1}{x^3 z^2} + \frac{2}{x^2 z} + \cancel{1} + \frac{1}{xz} - \cancel{\frac{1}{x}}$$

$$-\frac{z'}{z^2} = \left(\frac{2}{x^2} + \frac{1}{x}\right)\frac{1}{z} + \frac{1}{x^3 z^2}$$ Multiply by $-z^2$

$$z' = -\left(\frac{2}{x^2} + \frac{1}{x}\right) z - \frac{1}{x^3}$$

$$z' + \left(\frac{2}{x^2} + \frac{1}{x}\right) z = -\frac{1}{x^3}$$ $\longrightarrow$ linear 1st order ODE.

$$\mu(x) = e^{\int \frac{2}{x^2} + \frac{1}{x} dx} = e^{-\frac{2}{x} + \ln x} = e^{-\frac{2}{x}} \cdot e^{\ln x} = x e^{-\frac{2}{x}}$$

is an I.F.

Multiply the last equation by $\mu(x) = x e^{-\frac{2}{x}}$ to get

$$x e^{-\frac{2}{x}} z' + \left(\frac{2}{x^3} + \frac{1}{x}\right) x e^{-\frac{2}{x}} z = -\frac{1}{x^2} e^{-\frac{2}{x}}$$

$$\left(x e^{-\frac{2}{x}} z\right)' = -\frac{1}{x^2} e^{-\frac{2}{x}}$$

$$(x e^{x} z) = -\frac{1}{x^2} e$$

$$x e^{-\frac{2}{x}} z = \int -\frac{1}{x^2} e^{-\frac{2}{x}} \, dx \qquad u = -\frac{2}{x} \quad du = \frac{2}{x^2} dx$$

$$= -\frac{1}{2} \int e^{u} \, du = -\frac{1}{2} e^{u} + c$$

$$= -\frac{1}{2} e^{-\frac{2}{x}} + c = \frac{-e^{-2x} + 2c}{2}$$

$$z = \frac{-e^{-2/x} + 2c}{2 x e^{-2/x}} = \frac{-1 + 2ce^{\frac{2}{x}}}{2x}$$

$$y = x + \frac{1}{z} \Rightarrow \boxed{y = x + \frac{2x}{-1 + 2ce^{2/x}}}$$

c) $y(1) = -1$

$$\Rightarrow -1 = 1 + \frac{2}{-1 + 2ce^{2}}$$

$$\Rightarrow -2 = \frac{2}{-1 + 2ce^{2}} \Rightarrow -1 + 2ce^{2} = -1$$

$$\Rightarrow c = 0$$

$$y = x + \frac{2x}{-1+0} \Rightarrow y = x - 2x \qquad y = -x.$$

3) Consider Riccati equation

$$y' = -\frac{1}{x(x-1)}y^2 + \frac{2x+1}{x(x-1)}y - \frac{2}{x-1}$$

a) Verify that $f(x) = x$ satisfies the given equation

b) Find the general solution of the given equation.

a) $$f' = -\frac{1}{x(x-1)}f^2 + \frac{2x+1}{x(x-1)}f - \frac{2}{x-1}$$

$$f(x) = x \Rightarrow f' = 1$$

$$-\frac{1}{x(x-1)}f^2 + \frac{2x+1}{x(x-1)}f - \frac{2}{x-1} = -\frac{1}{x(x-1)}x^2 + \frac{2x+1}{x(x-1)}\cdot x - \frac{2}{x-1}\left(\frac{x}{x}\right)$$

$$= \frac{-x^2}{x(x-1)} + \frac{2x^2+x}{x(x-1)} - \frac{2x}{x(x-1)} = \frac{-x^2+2x^2+x-2x}{x(x-1)}$$

$$= \frac{x^2-x}{x(x-1)} = 1$$

b) $y = f(x) + \frac{1}{z} = x + \frac{1}{z}$ put in the eqn

$$\left(x+\frac{1}{z}\right)' = -\frac{1}{x(x-1)}\left(x+\frac{1}{z}\right)^2 + \frac{2x+1}{x(x-1)}\left(x+\frac{1}{z}\right) - \frac{2}{x-1}$$

$$1 - \frac{z'}{z^2} = -\frac{1}{x(x-1)}\left(x^2 + \frac{1}{z^2} + \frac{2x}{z}\right) + \frac{2x^2+x}{x(x-1)} + \frac{2x+1}{x(x-1)}\cdot\frac{1}{z} - \frac{2}{x-1}$$

$$\cancel{1} - \frac{z'}{z^2} = -\cancel{\frac{x^2}{x(x-1)}} - \frac{1}{x(x-1)}\cdot\frac{1}{z^2} - \frac{2x}{x(x-1)}\cdot\frac{1}{z} + \cancel{\frac{2x^2+x}{x(x-1)}} + \frac{2x+1}{x(x-1)}\cdot\frac{1}{z} - \cancel{\frac{2}{x-1}}$$

$$-\frac{z'}{z^2} = -\frac{1}{x(x-1)}\cdot\frac{1}{z^2} + \frac{1}{x(x-1)}\cdot\frac{1}{z}$$

Multiply by $-z^2$

$$z' = \frac{1}{x(x-1)} - \frac{1}{x(x-1)}z$$

1st order ODE

$$z' = \frac{1}{x(x-1)} - \frac{1}{x(x-1)} z$$

$$z' + \frac{1}{x(x-1)} z = \frac{1}{x(x-1)} \rightarrow \text{a linear } \underbrace{\text{ODE}}_{} \quad \text{1st order}$$

$$\mu(x) = e^{\int \frac{1}{x(x-1)} dx}$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{Ax - A + Bx}{x(x-1)}$$

$$Ax + Bx - A = 1$$
$$(A+B)x - A = 1$$
$$A + B = 0$$
$$-A = 1$$

$$A = -1 \qquad B = 1$$

$$\mu(x) = e^{\int \frac{1}{x-1} - \frac{1}{x} dx} = e^{\ln(x-1) - \ln x} = e^{\ln\left(\frac{x-1}{x}\right)}$$
$$= \frac{x-1}{x} \checkmark$$

Multiply the last eqn by $\frac{x-1}{x}$

$$\frac{x-1}{x} z' + \frac{1}{x^2} z = \frac{1}{x^2}$$

$$\left(\frac{x-1}{x} z\right)' = \frac{1}{x^2} \Rightarrow \frac{x-1}{x} z = \int \frac{1}{x^2} dx$$

$$\frac{x-1}{x} z = -\frac{1}{x} + c \Rightarrow \frac{x-1}{x} z = \frac{-1 + cx}{x}$$

$$z = \frac{cx - 1}{x - 1}$$

$$y = x + \frac{1}{z} = x + \frac{x-1}{cx-1} \checkmark$$